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1 Introduction

Synchronization is an extremely important and interesting emergent property of complex systems. The first example found in literature goes back to the 17th century with Christiaan Huygens works [8, 1]. This kind of emergent behavior can be found in artificial systems as well as in natural ones and at many scales (from cell to whole ecological systems). Biology abounds of periodic and synchronized phenomena and the works of Ilya Prigogine showed that such behaviors arise within specific conditions: a dissipative structure generally associated to a non-linear dynamic [12]. Biological systems are open, they evolve far from thermodynamic equilibrium and are subject to numerous regulating processes, leading to highly non-linear dynamics. Therefore periodic behaviors appear (with or without synchronization) at any scale [13]. More generally, life itself is governed by circadian rhythms [6]. Those phenomena are as much attractive as they are often spectacular: from cicada populations that appear spontaneously every ten or thirteen years [7] or networks of heart cells that beat together [9] to huge swarms in which fireflies, gathered in a same tree, flash simultaneously. Furthermore, beyond biology one can find a wide source of examples in completely different fields of science (e.g. in behavioral psychology with the example of synchronizing applause [15]). For much more artificial and/or theoretical examples, one can consider the whole field of research that studies the coupling of smooth dynamical systems. Nowadays, it is one of the most important subject related to non-linear systems' dynamics, especially through the notion of chaotic systems' synchronization [10, 5]. This wide source of examples leads the field of research to be highly interdisciplinary, from pure theory to concrete applications and experimentations. The classical concept of synchronization is related to the locking of the basic frequencies and instantaneous phases of regular oscillations. Those questions are usually addressed by studying specific kinds of coupled discrete or differential systems, using classical tools of the field (see for instance [11]). Convinced that synchronization phenomenon is completely natural in a large

variety of coupled dynamical systems, we propose a new approach of the subject: firstly, we ask the question of synchronization differently than the usual way. Rather than trying to prove that synchronization actually takes place, we search conditions under which frequencies are locked as soon as the whole system oscillate. Secondly we enlarge the scope of handled models, by building a general framework for coupled systems called "tissular coupling". This framework is inspired by biological observations at cell's scale, but relevant at any scale of modeling. Under some general assumptions on the kind of interactions that constitute the coupling of the systems, we prove that for a wide class of tissular coupling systems, frequencies are mutually locked to a single value as soon as the whole population is oscillating.

This paper exhibits our model of tissular coupling and the frequency locking in the case of a finite number of coupled systems. In the first section we present some mathematical tools and the background we have used in order to study synchronization issue (the results exposed at the end of this paper is only a part of what we have fulfilled, and surely a really small part of what can be done using tissular coupling, this is why we state this framework in its general form). Then, we describe dynamical objects on which we focus, namely the tissular coupling and periodical motions of a population. In the second section we expose a useful way to reduce the problem to a structural one, with no more reference to the dynamics of the coupled systems. In the final section we exhibit some natural conditions under which we are able to prove the main result of this paper, a case of synchronization, in terms of frequencies locking. In a second paper [3] we expose the case of an infinite compact and connected population, which is processed with different mathematical tools.

2 Basic material and notation

As our model is inspired by cellular tissues, several terms clearly come from those kinds of complex systems. Nevertheless, its scope is not limited to cells nor cell tissues.

Model of population

Here are the basic compounds of our model:

- a **population** $\mathcal{P}$ is a measured space with finite measure. Typical populations are finite populations with counting measure or measurable subset of $\mathbb{R}^m$ with Lebesgue measure. Every $p \in \mathcal{P}$ is called a cell;
- for sake of concision, we suppose that the systems we want to study are $\mathbb{R}^n$-valued. Thus, a **state** of $\mathcal{P}$ is an element of $\mathcal{S} = \mathcal{M}_b(\mathcal{P}, \mathbb{R}^n)$, the space of measurable bounded applications from $\mathcal{P}$ to $\mathbb{R}^n$. A state of $\mathcal{P}$ will be denoted s, $s(p)$ representing the state of the cell p;

- let I be a real interval and $r \in \mathbb{N}$. A **trajectory** of $\mathcal{P}$ is an element of $\mathcal{F}(I, \mathcal{S})$. For sake of concision, any trajectory will be written s again, and $s(t,p)$ stands for the state of cell p at time t (which should be written $s(t)(p)$). We then define the space of trajectories as:

$$\mathcal{T}^r = \left\{ \mathcal{C}^r(I, \mathcal{S}),\ \forall \text{ compact } J \subset I,\ \sup_{J \times \mathcal{P}} \|s(t,p)\| < \infty \right\}$$

  A trajectory s is regular only along the time t (excepted in terms of measurability), thus we use the following unambiguous notation :

$$s'(t,p) = \partial_1 s(t,p)$$

- as there is no ambiguity, all norms will be denoted in the same manner :

$$\begin{aligned} s \in \mathcal{S} :\ \|s\| &= \sup_{\mathcal{P}} \|s(p)\| \\ s \in \mathcal{T} :\ \|s\| &= \sup_{I \times \mathcal{P}} \|s(p,t)\| \\ \|s(t,.)\| &= \sup_{\mathcal{P}} \|s(t,p)\| \\ \|s(.,p)\| &= \sup_{I} \|s(t,p)\| \end{aligned}$$

  With this kind of norm, $\mathcal{T}^0$ becomes obviously a Banach space.

Periodic motions and properties of periods on $\mathcal{P}$

Now, we define the notion of periodic trajectories on a population and some mathematical tools related to their study.

Definition 1. *A **period on** $\mathcal{P}$ is a map τ from $\mathcal{P}$ to $\mathbb{R}_+^*$. A trajectory $s \in \mathcal{T}^r$ is said to be τ-**periodic** if for any $p \in \mathcal{P}$, $s(.,p)$ is $\tau(p)$-periodic. $\tau(p)$ is then called the **period of** p. The space of such trajectories is written $\mathcal{T}_\tau^r$*

For any $s \in \mathcal{T}_\tau^0$ we define the τ-mean $I^\tau(s)$ of s as:

$$I^\tau(s)(p) = \frac{1}{\tau(p)} \int_0^{\tau(p)} s(t,p)dt$$

Note that this quantity is $\mathbb{R}^n$-valued.

Here follows a corollary of the ergodic theorem [4] (and for references to ergodic theory see for instance [14, 2]):

Proposition 1 (Ergodic alternative). *Let f a smooth real-valued map, τ-periodic an $\tau' > 0$. We define the τ'-average of f:*

$$a_{\tau'}(f)(t) = \lim_{N \to +\infty} \frac{1}{N+1} \sum_{k=0}^{N} f(t + k\tau')$$

Then the following alternative holds:

- *τ and τ' are $\mathbb{Z}$-free (non commensurable), then $a_{\tau'}(f)$ is constant and equals $\tau I^{\tau}(f)$;*
- *there's an irreducible fraction $\frac{a}{b}$ such that $b\tau = a\tau'$, then $a_{\tau'}(f)$ is $\frac{\tau'}{b}$-periodic.*

The synchronization issues we want to study are related to the following question: as soon as the whole system oscillates, which assumptions insure that cells oscillate according to the same period. We will see that showing that periods are commensurable is the first natural step to achieve an answer to this question. We need the following notion: a (finite) subset $\{\tau_1, \ldots, \tau_k\}$ of $\mathbb{R}$ is said to be **dependent** (over $\mathbb{Z}$) if there exists some integers $l_1, \ldots, l_k$ non all zeros and such that:

$$l_1\tau_1 + \ldots + l_k\tau_k = 0$$

Let's go back to differential systems on $\mathcal{P}$ now to complete our description of their coupling.

Differential systems on $\mathcal{P}$

As we want to reach the most general result, each cell p is supposed to behave according to an autonomous differential system given by an application F_p. Considering non-autonomous differential systems could be assimilated to the case of synchronization by an external force. Moreover, as we work with any kind of population, this external force could be assimilated to new cells and integrated in the population. For sake of concision, we will assume that F_p is smooth. Then, we have a family of applications $\{F_p\}_{p\in\mathcal{P}}$. We can define the extension $F_{\mathcal{P}}$ of this family to $\mathcal{S}$:

$$\begin{aligned} F_{\mathcal{P}} : \mathcal{S} &\longrightarrow \mathcal{S} \\ s &\longmapsto F_{\mathcal{P}}(s) \end{aligned}$$

with

$$F_{\mathcal{P}}(s)(p) = F_p(s(p))$$

The extension $F_{\mathcal{P}}$ is said to be $\mathcal{C}^r$ if every F_p is $\mathcal{C}^r$ and if for any bounded subset B of $\mathbb{R}^n$ we have:

$$\sup_{x\in B\times p\in\mathcal{P}} \|dF_p(x)\| < \infty$$

Coupler on $\mathcal{P}$

Now, we consider how to couple the differential systems given by $F_{\mathcal{P}}$. Usually [11], the coupling is made by adding to $s'(t,p)$ a quantity that depends on $s(t,.)$:

$$s'(t,p) = F_p(s(t,p)) + c(s(t,.))(p) \tag{1}$$

where $c(s(t,.))$, an element of $\mathcal{S}$, is a function of the global state of $\mathcal{P}$. This is what we naturally name a coupling **in the final space**. Another way of coupling is **in the initial space** :

$$s'(t,p) = F_{\mathcal{P}}\Big(c\big(s(t,.)\big)\Big)(p)$$

Coupling in the final space or in the initial space may have different interpretations. The first coupling method could be seen as physical exchanges and the second one as an instantaneous exchanges (for instance an high speed information exchange which could be modeled as instantaneous with regards to the system time scale). But, in most situations, those two couplings are equivalent. Indeed, the right-hand-side term of the equation (1) has the following shape:

$$F(x) + \varepsilon$$

so that, as soon as this r.h.s. stays in the range of the map F, one can find a vector y:

$$F(x) + \varepsilon = F(y)$$

writing $\delta = y - x$ we have:

$$F(x) + \varepsilon = F(x + \delta)$$

In fact, this case is quite general as ε is usually small, and even if it is not, we will study only periodical trajectories occurring essentially around limit cycles surrounded by the range of F. For these reasons, we will concentrate on the last kind of coupling. Now we give technical details about our method of coupling, by the way of a **coupler** :

$$\begin{aligned} C : \mathcal{S} &\longrightarrow \mathcal{S} \\ s &\longmapsto C(s) \end{aligned}$$

This coupler is said to be **linear** if there is an application c satisfying:

$$\begin{aligned} c : \mathbb{R}^n \times \mathcal{P} &\longrightarrow \mathcal{L}\left(\mathcal{S}, \mathbb{R}^n\right) \\ (x,p) &\longmapsto c(x,p) \end{aligned}$$

such that for all $p \in \mathcal{P}$:

$$C(s)(p) = c\left(s(p), p\right).s$$

In other words, depending on the cell p (*e.g.* its spatial position) and on its state $s(p)$, the coupling compute linearly (as a mean) the pseudo-state $c\left(s(p), p\right).s$ dictating the evolution of this cell, instead of the state $s(p)$ which is used if the cell is isolated (without coupling).
Sometimes the following point of view, the operator one, is be useful:

$$\begin{aligned} L_c : \mathcal{S} &\longrightarrow \mathcal{L}(\mathcal{S}, \mathcal{S}) \\ s &\longmapsto L_c(s) \end{aligned}$$

where:

$$[L_c(s).s](p) = c\left(s(p), p\right).s$$

If L_c is constant, the coupler is said to be **uniformly linear** and is naturally associated to an element of $\mathcal{L}(\mathcal{S}, \mathcal{S})$. The case of uniformly linear coupler is the simpler one: the way a cell is influenced by its environment does not depends on its proper state. It is an approximation of what happens in real systems, but this simplification allows us to reach very global results.

To familiarize with the notion of coupler, let's write how it transforms a trajectory of $\mathcal{P}$. If $s \in \mathcal{T}$, the effect of a coupler on this trajectory is:

$$C(s)(t,p) = c(s(t,p),p).s(t,.)$$

Tissular coupling

We are now able to define a tissular coupler. Given a population $\mathcal{P}$, a coupler C is said to be a **tissular coupler** if the associated application c satisfies:

$$c(x,p).s = \int_{\mathcal{P}} c_d(x,p,q).s(q)dq + c_a(x,p).s(p)$$

where c_d and c_a are continuous and are called diffuse part and atomic part of c (and of C), respectively. The diffuse part represents the influence of the environment (the whole population) on the behavior of the cell p. The atomic part models the self-influence of p. We could have merge this into an unique linear operator, containing a Dirac impulse, but for both comprehension and manipulation purposes we think that this shape is preferable and less theoretical.

Finally, we can define a tissular coupling by specifying the equation that any solution should satisfy. Given a population $\mathcal{P}$, a family of applications $\{F_p\}_{\mathcal{P}}$ and a tissular coupler C, the **tissular coupling** $S(\mathcal{P}, F_{\mathcal{P}}, C)$ is defined by the equation:

$$s' = F_{\mathcal{P}} \circ C(s) \tag{2}$$

in other words, any solution s satisfies for all $(t,p) \in I \times \mathcal{P}$:

$$s'(t,p) = \int_{\mathcal{P}} c_d(s(t,p),p,q).s(t,q)dq + c_a(s(t,p),p).s(t,p)$$

We naturally end this introduction of the tissular coupling with a result on the existence of solutions, which is proved in a classical way [4]:

Theorem 1. *Suppose $F_{\mathcal{P}}$ is $\mathcal{C}^1$ and c_d and c_a are locally lipschtizian in their first variable. Given any initial condition (t_0, s_0) in $I \times \mathcal{S}$, the tissular coupling given by equation (2) admits a unique maximal solution.*

We can then go further and begin to work on the heart of our matter.

3 Reduction of the synchronization problem

Cyclic application

The first reduction of the problem we have done was deciding to handle coupling in the initial space. Indeed, by the way of a natural assumption made on $F_{\mathcal{P}}$ we will reduce the problem to a structural one.

Definition 2. *Let F be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and U an open subset of $\mathbb{R}^n$. F is said to be cyclic on U if for any motion $x : \mathbb{R} \to U$ such that there exists τ satisfying:*

$$F \circ x(t+\tau) = F \circ x(t) \quad \forall t \in \mathbb{R}$$

then:

$$x(t+\tau) = x(t) \quad \forall t \in \mathbb{R}$$

Before we expose what the problem of frequency locking becomes under this assumption, we should explain why considering cyclic maps is somewhat natural. For example, if one tries to build an elementary non-cyclic map, one can consider the following:

$$\begin{aligned} F : \mathbb{R}^2 - \{0\} &\to \mathbb{R}^2 - \{0\} \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\mapsto \frac{1}{\|x\|} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \end{aligned}$$

for which any map $x(t)$ satisfying :

$$x(t) = \begin{bmatrix} r(t)\cos(t) \\ r(t)\sin(t) \end{bmatrix}$$

leads to (fig. 1):

$$F \circ x(t+\omega) = F \circ x(t)$$

But this kind of applications is not of great interest in the scope of physical and biological phenomena, indeed if we want to consider a little more credible version of this map, we must look at the mathematical pendulum, which is defined by:

$$\begin{aligned} F : \mathbb{R}^2 - \{0\} &\to \mathbb{R}^2 - \{0\} \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\mapsto \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \end{aligned}$$

for which the preceding property does not hold any more (it's a one-to-one map). Moreover, even if the map isn't one-to-one, in a realistic model of a physical phenomenon, periodic trajectories are rare: given such a dynamical system, any specific periodic trajectory in the phase space corresponds to a limit cycle. Those limit cycles are indeed rare in non conservative systems (compare for instance the mathematical and the physical pendulum). In other words, there exists a neighborhood of those limit cycles in which the only maps $x(t)$ such that $F \circ x(t)$ is periodic are precisely the limit cycles themselves,

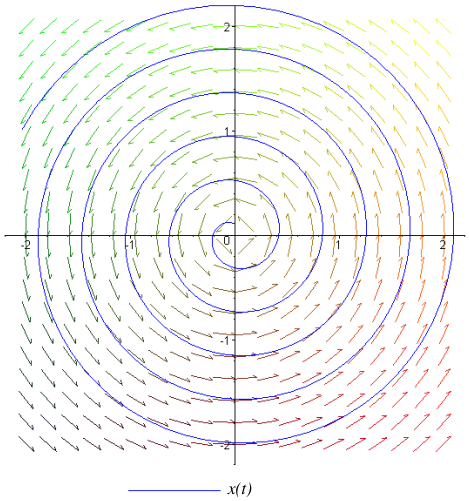

Fig. 1. Vector field associated to F and an example of non periodic map $x(t)$ such that $F \circ x$ is periodic.

which are effectively periodic applications.
Moreover, this condition of cyclicity is extremely useful and allows us to prove the following result:

Proposition 2. *Let $(\mathcal{P}, C, F_{\mathcal{P}})$ be a tissular coupling and τ a period on $\mathcal{P}$. Let's assume that there exists a family $\{U_p\}_{p\in\mathcal{P}}$ of open subsets of $\mathbb{R}^n$ such that for all $p \in \mathcal{P}$ F_p is cyclic on U_p. If s is a τ-periodic solution of the coupling such that for all $p \in \mathcal{P}$, $s(I, p) \subset U_p$, then s belongs to the following set:*

$$A(\tau, C) = \mathcal{T}^\infty \cup C^{-1}(\mathcal{T}^\infty)$$

We then have drastically reduced the problem: under hypotheses of the proposition 2 we can throw $F_{\mathcal{P}}$ and only work with the coupler C and the space of periodic solutions.
This last section exhibits the main results of our work on finite populations.

4 Synchronization in finite population

Here we focus on a coupler in the case of a finite population $\mathcal{P}$:

$$\mathcal{P} = \{p_1, \ldots, p_k\}$$

If C is a tissular coupling, the diffuse part is sufficient to define it, thus we can also write it for $x \in \mathbb{R}^n$ as:

$$c_d(x, p_i, pj) = c_{ij}(x) \in \mathcal{L}(R^n)$$

L_c adopts naturally a matrix shape:

$$L_c(x) = \begin{bmatrix} c_{11}(x) & \dots & c_{1k}(x) \\ \vdots & \ddots & \vdots \\ c_{k1}(x) & \dots & c_{kk}(x) \end{bmatrix}$$

We point out that each $c_{ij}(x)$ is a linear application, an element of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ (to be precise and to anticipate on some theoretical generalizations of the tissular coupling, we note that $L_c(x)$ is the matrix of a linear application on $(\mathbb{R}^n)^k$ seen as a module on the ring $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$).
The image of the motion $s(., p_i)$ is thus given by:

$$C(s)(t, p_i) = \sum_{j=1}^{k} c_{ij}(s(t, p_i))s(t, p_j)$$

The kind of assumptions we are about to make on C involve linear properties. Especially we work with the rank of L_c and of sub-matrices. Thus, if $J \subset \{1, \cdots, k\}$ contains l elements, we write $I^c = \{1, \cdots, k\} - J$. If $M = (m_{ij})$ is an $k \times k$ matrix, we define M^I as the $l \times (l - k)$ matrix which coefficients are the $m_{i,j}$ for $(i, j) \in I \times I^c$. For $s \in \mathcal{S}$ or $\mathcal{T}$ we write s^I the vector $[s(p_i)]_{i \in I}$ or $[s(., p_i)]_{i \in I}$ (see figure 2).

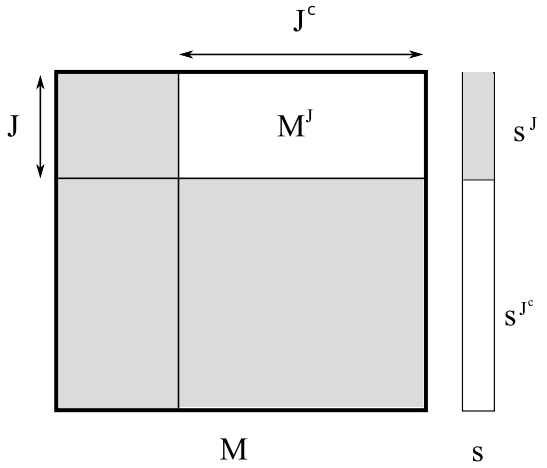

Fig. 2. Matrix and vectors associated to a subset J of $\{1, \cdots, k\}$.

Definition 3. *Let C a coupler on $\mathcal{P}$. C is said to be **reducible** if there exists $x \in \mathbb{R}^n$ and a non trivial part J of $\{1, \cdots, n\}$ such that $L_c^J(x)$ has non-invertible coefficients (as elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$). C is **non-reducible** if it's not reducible.*

The next step consists in "cleaning" the space of periodic solutions in order to exclude some degenerated and/or trivial cases. First the trivial periodic solutions:

Definition 4. *Any s in $A(\tau, C)$ is said to be non trivial if for all $p \in \mathcal{P}$, neither $s(.,p)$ and $c(e)(.,p)$ are constant maps.*

Now we consider some trajectories that are degenerated in a stronger sense than the one of trivial trajectories: our aim is to avoid periodic trajectories which oscillating behaviors are the only consequences of the coupler's dynamic (we focus on the synchronization of interacting oscillating systems, not on some systems oscillating due to an external force, and we have already mentioned that this kind of external force can be included in the population itself). Such trajectories are those for which there exists a state b satisfying:

$$L_c\big(s(t,.)\big)\big(s(t,.) - b\big) = 0$$

In terms of matrices, we then have for the cell p_i :

$$\sum_{j=1}^{k} c_{ij}\big(s(t,p_i)\big)\big(s(t,p_j) - b(p_j)\big) = 0$$

In other words, the preceding equality shows that the vector $[s(.,p_1), \cdots, s(.,p_k)]$ is not linearly free, according to the compounds of $L_c\big(e(t,.)\big)$. These observations lead us to the following definition:

Definition 5. *Let $\mathcal{P}$ be a finite population, C a tissular coupler on $\mathcal{P}$ and τ a period on $\mathcal{P}$. A non-trivial element $s \in A(\tau, C)$ is said to be C-**free** if for any $b \in \mathcal{S}$ and for any non-trivial $J \subset \{1, \cdots, k\}$ we have:*

$$L_c\big(s(t,.)\big)^J.\big(s(t,.) - b\big)^{J^c} \neq 0$$

In fact, we can prove that, when C is non-reducible, a non C-free element is degenerated because it comes from a solution of a system on a population which cardinal is strictly lower than k.

Here is the first step to frequencies locking as it deals with dependency of periods [4]:

Theorem 2. *If C is non-reducible and if $A(\tau, C)$ contains a non-trivial and C-free element, then $\tau(\mathcal{P})$ is a dependent set.*

Finally we extend this result to a more powerful conclusion. In order to prove synchronization, we must add a last condition on C. This is done introducing the following definition:

Definition 6. *Let C be a uniformly linear coupler on $\mathcal{P}$. C is said to be strongly non-reducible if it is non-reducible and if for all $x \in \mathbb{R}^n$ and all non trivial subset $J \subset \{1, \ldots, k\}$ with more than $\frac{k}{2}$ elements, $L_c^J(x)$ is injective.*

Remark 1. Because of its dimensions, $L_c^J(x)$ can't be injective if J contains less than $\frac{k}{2}$ elements.

We can now state our main result [4]:

Theorem 3. *Let $\mathcal{P}$ be a finite population of cardinal k, C a tissular coupler, uniformly linear and strongly non-reducible and τ a period on $\mathcal{P}$. Suppose $s \in A(\tau, C)$ isn't trivial and is non-reducible. If more than $\frac{k}{2}$ cells have the same period then all cells have the same period.*

5 Conclusion

In this work we have built a general framework, the tissular coupling, to handle a wide variety of coupled systems, and therefore a wide class of complex systems. We focused on an emergent property of those dynamical systems: synchronization, and precisely frequencies locking. We used the notion of tissular coupling to show that the synchronization issue may be addressed differently. Usually one observes solutions of particular coupled systems and shows that within suitable conditions synchronization must occur. Those results are often qualitatively dependent of the studied systems and do not stand in the general case. We tried to change our point of view and to bring completing results. These are less precise than usual ones as we don't prove that synchronization ultimately happens. Instead, we consider the problem at its end: if one supposes that coupled systems oscillate, then they must be synchronized. The loss in time evolution informations is compensated by very general results, almost independent from the individual differential systems to be coupled. We believe that this complementary approach of frequencies locking will lead to future developments as it brings about many unused mathematical tools. This will be showed in a following paper in which we study the same question concerning an infinite compact connected populations but with very different methods [3]. Other examples of the use of tissular coupling can be found in [4] on emergence of spatial patterns. All those work and results are stimulating and seem to prove that this framework will be fruitful.

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